

Two reconstruction procedures for a 3-d phaseless inverse scattering problem for the generalized Helmholtz equation

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Abstract

The 3-d inverse scattering problem of the reconstruction of the unknown dielectric permittivity in the generalized Helmholtz equation is considered. The main difference with the conventional inverse scattering problems is that only the modulus of the scattering wave field is measured. The phase is not measured. The initializing wave field is the incident plane wave. On the other hand, in the previous recent works of the authors about the “phaseless topic” the case of the point source was considered [20, 21, 22]. Two reconstruction procedures are developed for a linearized case. However, the linearization is not the Born approximation. This means that, unlike the Born approximation, our linearization does not break down when the frequency tends to the infinity. Applications are in imaging of nanostructures and biological cells.

Keywords: phaseless inverse scattering problem, generalized Helmholtz equation, reconstruction formula, Radon transform

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1 Introduction

This paper is the continuation of three recent publications [20, 21, 22] of the authors dedicated to the reconstruction procedures for 3-d Phaseless Inverse Scattering Problems (PISPs). Our goal here is to extend the result of [22] to the case when the wave propagation process is generated by an incident plane wave rather than by the point source of [22]. We end up our derivation with the inversion of the 2-d Radon transform. This can be done by the well known technique, see, e.g. [26]. We also propose a different inversion method based on the integral geometry. The idea of the second method goes back to the famous paper of Allan Cormack published in 1963 [6]. It is well known that his publication [6] led him to the Nobel prize in 1979, along with Godfrey Hounsfield, for his work on X-ray computed tomography (CT), see https://en.wikipedia.org/wiki/Allan_McLeod_Cormack.

In [20] a reconstruction procedure was proposed for a 3-d PISP for the Schrödinger equation in the frequency domain. The next question is about the reconstruction for the generalized Helmholtz equation. An important difference between these two is that in the generalized Helmholtz equation the unknown coefficient is multiplied k^2 , where $k > 0$ is the frequency, and k is varied in our problems. On the other hand, in the Schrödinger equation the unknown potential is not multiplied by k^2 . This makes the reconstruction procedure for the PISP for the Schrödinger equation easier than that for the generalized Helmholtz equation.

In [21] the reconstruction procedure was developed for the case when the Born approximation is considered for the generalized Helmholtz equation. However linearization of the Born approximation breaks down for large values of k . Hence, in [22] the reconstruction procedure for that equation was developed for the case when the wave propagation process is generated by the point source. Even though a linearization was used in [22], this linearization does not break down for large k . Still, we point out that both here and in [22] the phase of the scattered wave field is reconstructed without the linearization, separately for each direction of the incident wave field. The only approximation we use when reconstructing the phase is that we ignore the term $O(1/k)$ for $k \rightarrow \infty$.

Inverse scattering problems without the phase information arise in imaging of structures whose sizes are of the micron range or less. Recall that 1 micron ($1\mu m$) = $10^{-6}m$. For example, nano structures typically have sizes of hundreds on nanometers, which is about $0.1\mu m$. Therefore, the wavelength for this imaging should be about $0.1\mu m$. The second example is in imaging of living biological cells. Sizes of cells are between $5\mu m$ and $100\mu m$ [33, 34]. Either optical or X-ray radiation is used in this imaging. It is well known that for the micron range of wavelengths only the intensity of the scattered wave field can be measured, and the phase cannot be measured [7, 8, 14, 32, 40]. The intensity is the square modulus of the scattered complex valued wave field. We point out that standard statements of inverse scattering problems assume that both the modulus and the phase of the complex valued scattered wave field can be measured, see, e.g. [10, 11, 25, 28, 29].

As to the uniqueness results PISPs, see [1, 16, 27] for the 1-d case and [17, 18, 19] for the 3-d case. However, proofs of these results are not constructive. The first rigorous reconstruction procedures were proposed recently by the authors [20, 21, 22] and independently and for different statements of problems by Novikov [30, 31]. In [12, 13] numerical methods for reconstructions of shapes of obstacles from phaseless scattered data were developed. As to some heuristic reconstruction algorithms, we refer to, e.g. works of physicists [7, 8, 14, 32, 40].

In section 2 we state both forward and inverse problems. In section 3 we study geodesics lines generated by the refractive index $n(x), x \in \mathbb{R}^3$. In section 4 we study an auxiliary Cauchy problem for a hyperbolic equation. In section 5 we show how to reconstruct the phase for high frequencies. In section 6 we linearize the problem. Finally in section 7 we present two reconstruction procedures.

2 Statements of forward and inverse problems

Let $B > 0$ and $\Omega = \{|x| < B\} \subset \mathbb{R}^3$ be the ball of the radius B with the center at $\{x = 0\}$. Denote $S = \{|x| = B\}$. Let the refractive index $n(x), x \in \mathbb{R}^3$ be a function satisfying the

following conditions

$$n(x) \in C^{15}(\mathbb{R}^3), \quad n(x) = 1 + \beta(x), \quad (2.1)$$

$$\beta(x) \geq 0, \quad \beta(x) = 0 \quad \text{for } x \in \mathbb{R}^3 \setminus \Omega. \quad (2.2)$$

The C^{15} smoothness of $n(x)$ will be explained below (see the proof of Theorem 1). Let $S^2 = \{\nu \in \mathbb{R}^3 : |\nu| = 1\}$ be the unit sphere. We consider the following equation

$$\Delta u + k^2 n^2(x) u = 0, \quad x \in \mathbb{R}^3, \quad (2.3)$$

$u(x, k)$ is the complex valued wave field and $k > 0$ is a positive real number. We seek the solution of equation (2.3) in the form

$$u(x, k, \nu) = \exp(-ikx \cdot \nu) + u_{sc}(x, k, \nu), \quad (2.4)$$

where the function $u_{sc}(x, k, \nu)$ satisfies the radiation condition at the infinity,

$$\frac{\partial u_{sc}}{\partial r} + iku_{sc} = o(r^{-1}), \quad r = |x| \rightarrow \infty. \quad (2.5)$$

It is well known that the problem (2.3)-(2.5) has unique solution $u(x, k, \nu) \in C^3(\mathbb{R}^3)$, $\forall \nu \in S^2$ [5]. Furthermore, theorem 6.17 of [9] implies that, given (2.1), the function $u(x, k, \nu) \in C^{16+\alpha}(\mathbb{R}^3)$, $\forall \alpha \in (0, 1)$, $\forall \nu \in S^2$, where $C^{16+\alpha}(\mathbb{R}^3)$ is the Hölder space.

We model the propagation of the electric wave field in \mathbb{R}^3 by the solution of the problem (2.3)-(2.5). This modeling was numerically justified in [4] in the case of the time domain. It was shown numerically in [4] that this modeling can replace the modeling via the full Maxwell's system, provided that only a single component of the electric field is incident upon the medium. Then this component dominates two others and its propagation is well governed by the single equation (2.3). This conclusion was verified via accurate imaging using electromagnetic experimental data in, e.g. Chapter 5 of [3] and [41, 42].

We consider the following inverse problem:

Phaseless Inverse Scattering Problem. *Consider a number $k_0 > 0$. Find the function $\beta(x)$ assuming that the following function $f(x, \nu, k)$ is given*

$$f(x, \nu, k) = |u_{sc}(x, \nu, k)|^2, \quad \forall x \in S, \forall \nu \in S^2, \forall k \geq k_0, \quad (2.6)$$

3 Geodesic lines

Our reconstruction procedure is essentially using geodesic lines generated by the function $n(x)$. Hence, we consider these lines in this section. The Riemannian metric generated by the function $n(x)$ is

$$d\tau = n(x) |dx|, \quad |dx| = \sqrt{(dx_1)^2 + (dx_2)^2 + (dx_3)^2}. \quad (3.1)$$

For each vector $\nu \in S^2$ define the plane $\Sigma(\nu)$ as

$$\Sigma(\nu) = \{\xi \in \mathbb{R}^3 : \xi \cdot \nu = -B\}. \quad (3.2)$$

Let $\xi_0 \in \Sigma(\nu)$ be an arbitrary point. Then $(\xi - \xi_0) \cdot \nu = 0, \forall \xi \in \Sigma(\nu)$. Hence, the plane $\Sigma(\nu)$ is orthogonal to the vector ν . Also, note that

$$\Sigma(\nu) \cap \Omega = \emptyset. \quad (3.3)$$

We can represent an arbitrary point $\xi_0 \in \Sigma(\nu)$ as

$$\xi_0 = \xi_0(\eta_1, \eta_2) = -B\nu + e_1\eta_1 + e_2\eta_2, \quad (\eta_1, \eta_2) \in \mathbb{R}^2,$$

where unit vectors ν, e_1, e_2 form an orthogonal triple.

Let the function $\tau(\xi, \nu)$ be the solution of the Cauchy problem for the eikonal equation,

$$|\nabla_\xi \tau(\xi, \nu)| = n(\xi), \quad \tau|_{\Sigma(\nu)} = 0, \quad (3.4)$$

such that

$$\tau(\xi, \nu) \begin{cases} > 0 \text{ if } \xi \cdot \nu > -B, \\ < 0 \text{ if } \xi \cdot \nu < -B. \end{cases} \quad (3.5)$$

It is well known that $|\tau(\xi, \nu)|$ is the Riemannian distance between the point ξ and the plane $\Sigma(\nu)$. Physically, $|\tau(\xi, \nu)|$ is the travel time between ξ and the plane $\Sigma(\nu)$. For $\xi \cdot \nu < -B$ the function $\tau(\xi, \nu)$ has the form $\tau(\xi, \nu) = \xi \cdot \nu + B$. To find this function for $\xi \cdot \nu > -B$, we need to solve the problem (3.4) in this domain. It is well known that to solve this problem, one needs to solve a system of ordinary differential equations. These equations also define geodesic lines of the Riemannian metric. They are (see, e.g. [37]):

$$\frac{d\xi}{ds} = \frac{p}{n^2(\xi)}, \quad \frac{dp}{ds} = \nabla(\ln n(\xi)), \quad \frac{d\tau}{ds} = 1, \quad (3.6)$$

where s is a parameter and $p = \nabla_\xi \tau(\xi, \nu)$. Consider an arbitrary point $\xi_0(\eta_1, \eta_2) \in \Sigma(\nu)$ and the solution of the equations (3.6) with the Cauchy data

$$\xi|_{s=0} = \xi_0(\eta_1, \eta_2), \quad p|_{s=0} = n(\xi_0(\eta_1, \eta_2))\nu, \quad \tau|_{s=0} = 0. \quad (3.7)$$

Here the parameter $s \geq 0$. The solution of the problem (3.6), (3.7) defines the geodesic line which passes through the point $\xi_0(\eta_1, \eta_2)$ in the direction ν . Hence, this line intersects the plane $\Sigma(\nu)$ orthogonally. For $s > 0$ the solution determines the geodesic line $\xi = g(s, \eta_1, \eta_2, \nu)$ and the vector $p = h(s, \eta_1, \eta_2, \nu)$ which lays in the tangent direction to the geodesic line. It is well known from the theory of Ordinary Differential Equations that if the function $n(x) \in C^m(\mathbb{R}^3)$, $m \geq 2$, then functions g, h are C^{m-1} -smooth functions of their variables.

By (3.7)

$$\left. \frac{\partial \xi}{\partial \eta_1} \right|_{s=0} = e_1, \quad \left. \frac{\partial \xi}{\partial \eta_2} \right|_{s=0} = e_2.$$

Moreover, it follows from (3.6) and (3.7) that

$$\left. \frac{d\xi}{ds} \right|_{s=0} = \frac{\nu}{n(\xi_0(\eta_1, \eta_2))} = \nu.$$

Hence,

$$\left| \frac{\partial(\xi_1, \xi_2, \xi_3)}{\partial(s, \eta_1, \eta_2)} \right|_{s=0} = 1 \neq 0. \quad (3.8)$$

By (3.8) the equality $x = g(s, \eta_1, \eta_2, \nu)$ can be uniquely solved with respect to s, η_1, η_2 for those points x which are sufficiently close to the plane $\Sigma(\nu)$, as

$$s = s(x, \nu), \quad \eta_1 = \eta_1(x, \nu), \quad \eta_2 = \eta_2(x, \nu).$$

Hence, the equation

$$\xi = g(s, \eta_1(x, \nu), \eta_2(x, \nu), \nu) = \hat{g}(s, x, \nu), \quad s \in [0, s(x, \nu)]$$

defines the geodesic line $\Gamma(x, \nu)$ that passes through points x and $\xi_0(\eta_1(x, \nu), \eta_2(x, \nu)) = \xi_0(x, \nu)$. Extend $\Gamma(x, \nu)$ for $\xi \cdot \nu < -B$ by the equation $\xi = \xi_0(x, \nu) + s\nu$, $s < 0$. This line intersects the plane $\Sigma(\nu)$ at the point $\xi_0(x, \nu) \in \Sigma(\nu)$ orthogonally. The Riemannian distance between points x and $\xi_0(x, \nu)$ is $s(x, \nu) = \tau(x, \nu)$. Note that functions $\hat{g}(s, x, \nu)$ and $\hat{h}(s, x, \nu) = h(s, \eta_1(x, \nu), \eta_2(x, \nu), \nu)$ are C^{m-1} -smooth functions of their arguments. Since, moreover, $\hat{h}(s(x, \nu), x, \nu) = \nabla_x \tau(x, \nu)$, we conclude that $\tau(x, \nu)$ is C^m -smooth function. In our case $\tau(x, \nu)$ is C^{15} -smooth function of x for x enough closed to $\Sigma(\nu)$.

We have constructed geodesic lines $\Gamma(x, \nu)$ above only “locally”, i.e. only for those points x which are located sufficiently close to the plane $\Sigma(\nu)$. However, we need to consider these lines “globally”. Hence, everywhere below we rely on the following Assumption:

Assumption. *We assume that geodesic lines of the metric (3.1) with conditions (3.4), (3.5) satisfy the regularity condition in \mathbb{R}^3 . In other words, for each vector $\nu \in S^2$ and for each point $x \in \mathbb{R}^3$ there exists a single geodesic line $\Gamma(x, \nu)$ connecting x with the plane $\Sigma(\nu)$ such that $\Gamma(x, \nu)$ intersects $\Sigma(\nu)$ orthogonally.*

It is well known from the Hadamard-Cartan theorem [2] that in any simply connected complete manifold with a non positive curvature each two points can be connected by a single geodesic line. The manifold (Ω, n) is called the manifold of a non positive curvature, if the section curvatures $K(x, \sigma) \leq 0$ for all $x \in \Omega$ and for all two-dimensional planes σ . A sufficient condition for $K(x, \sigma) \leq 0$ was derived in [39]

$$\sum_{i,j=1}^3 \frac{\partial^2 \ln n(x)}{\partial x_i \partial x_j} \xi_i \xi_j \geq 0, \quad \forall x \in \Omega, \quad \xi \in \mathbb{R}^3. \quad (3.9)$$

In our case if condition (3.9) holds for $x \in \Omega$, then it also holds for all $x \in \mathbb{R}^3$.

Our Assumption implies that for any $\nu \in S^2$ and for any $x \in \mathbb{R}^3$ there exists a unique pair $(\xi_0(x, \nu), s(x, \nu))$ with $\xi_0(x, \nu) \in \Sigma(\nu)$ such that the geodesic line $\Gamma(x, \nu)$ connects points x and $\xi_0(x, \nu)$ and intersects the plane $\Sigma(\nu)$ orthogonally. Hence, there exists a function $\bar{g}(s, x, \nu)$ such that the equation of the geodesic line $\Gamma(x, \nu)$ is given by

$$\Gamma(x, \nu) = \{\xi : \xi = \bar{g}(s, x, \nu), s \in [0, s(x, \nu)]\}$$

and $x = \bar{g}(s(x, \nu), x, \nu)$. Then $s(x, \nu) = \tau(x, \nu)$ is the Riemannian length of $\Gamma(x, \nu)$. Also, $\hat{p}(s(x, \nu), x, \nu) = \nabla_x \tau(x, \nu)$ and $|\nabla_x \tau(x, \nu)| = n(x)$.

4 Auxiliary Cauchy problem for a hyperbolic equation

Modifying ideas of [20, 21, 22], we consider in this section an auxiliary Cauchy problem for a hyperbolic equation with the incident plane wave. Consider the hyperbolic equation

$$n^2(x)v_{tt} - \Delta v = 0, \quad x \in \mathbb{R}^3, t \in \mathbb{R}. \quad (4.1)$$

We seek the solution $v(x, t, \nu)$ of equation (4.1) in the form

$$v(x, t, \nu) = \delta(t - x \cdot \nu) + \bar{v}(x, t, \nu), \quad (4.2)$$

where $\delta(t - x \cdot \nu)$ is the incident plane wave propagating in the direction ν , and the function $\bar{v}(x, t, \nu)$ satisfies the following initial conditions

$$\bar{v}|_{t < -B} \equiv 0. \quad (4.3)$$

Let $T > 0$ be an arbitrary number. Let $\varphi(x, \nu)$ be the solution of the following problem

$$|\nabla_x \varphi(x, \nu)|^2 = n^2(x), \quad (4.4)$$

$$\varphi(x, \nu) = x \cdot \nu \text{ for all } x \cdot \nu \leq -B. \quad (4.5)$$

Let the function $A(x, \nu)$ be

$$A(x, \nu) = \begin{cases} \exp\left(-\frac{1}{2} \int_{\Gamma(x, \nu)} n^{-2}(\xi) \Delta_\xi \varphi(\xi, \nu) d\tau\right), & x \cdot \nu > -B, \\ 1, & x \cdot \nu \leq -B. \end{cases} \quad (4.6)$$

The reason of the second line of (4.6) is (4.5) as well as the fact that $\Gamma(x, \nu)$ is the straight line for $x \cdot \nu \leq -B$. Let $T > 0$ be an arbitrary number. Consider the domain $D(\nu, T)$ defined as

$$D(\nu, T) = \{(x, t) \mid \max(-B, \varphi(x, \nu)) \leq t \leq T\}. \quad (4.7)$$

Let $H(t)$ be the Heaviside function,

$$H(t) = \begin{cases} 1, & t > 0, \\ 0, & t < 0. \end{cases}$$

Theorem 1. *Let conditions (2.1), (2.2) as well as Assumption hold. Let $\nu \in S^2$ be an arbitrary vector and T be an arbitrary number. Then there exists unique solution $v(x, t, \nu)$ of the problem (4.1)-(4.3) which can be represented in the form*

$$v(x, t, \nu) = A(x, \nu) \delta(t - \varphi(x, \nu)) + \widehat{v}(x, t, \nu) H(t - \varphi(x, \nu)), \quad x \in \mathbb{R}^3, t \in (-\infty, T), \quad (4.8)$$

where the function $A(x, \nu)$ is defined in (4.6), the function $\widehat{v}(x, t, \nu) \in C^2(D(\nu, T))$ and $\widehat{v}(x, t, \nu) = 0$ for $t < -B$.

Proof. Introduce functions $\theta_k(t)$ as

$$\theta_{-3}(t) = \delta''(t), \quad \theta_{-2}(t) = \delta'(t), \quad \theta_{-1}(t) = \delta(t), \quad \theta_k(t) = \frac{t^k}{k!} H(t), \quad k = 0, 1, 2, \dots$$

Observe that $\theta'_k(t) = \theta_{k-1}(t)$ for all $k \geq -2$. We seek the solution to the problem (4.1)-(4.3) in the form

$$v(x, t, \nu) = A(x, \nu) \theta_{-1}(t - \varphi(x, \nu)) + \sum_{k=0}^r a_k(x, \nu) \theta_k(t - \varphi(x, \nu)) + w_r(x, t, \nu) \quad (4.9)$$

for $x \in \mathbb{R}^3, t \in \mathbb{R}, r \geq 1$. Substituting representation (4.9) in (4.1), using $|\nabla_x \varphi(x, \nu)|^2 = n^2(x)$ and equating coefficients at $\theta_k(t)$, we obtain formulas for finding coefficients $A(x, \nu)$, $a_k(x, \nu)$:

$$\begin{aligned} 2\nabla A(x, \nu) \cdot \nabla \varphi(x, \nu) + A(x, \nu) \Delta \varphi(x, \nu) &= 0, \\ 2\nabla a_0(x, \nu) \cdot \nabla \varphi(x, \nu) + a_0(x, \nu) \Delta \varphi(x, \nu) &= \Delta A(x, \nu), \\ 2\nabla a_k(x, \nu) \cdot \nabla \varphi(x, \nu) + a_k(x, \nu) \Delta \varphi(x, \nu) &= \Delta a_{k-1}(x, \nu), \quad k \geq 1. \end{aligned} \quad (4.10)$$

Since by (4.2) and (4.3) $v(x, t, \nu) = \delta(t - x \cdot \nu)$ for $t < -B$, then we obtain

$$A(x, \nu) = 1, \quad \text{for } x \cdot \nu \leq -B, \quad (4.11)$$

$$\varphi(x, \nu) = x \cdot \nu, \quad \text{for } x \cdot \nu \leq -B, \quad (4.12)$$

$$a_k(x, \nu) = 0, \quad k \geq 0, \quad \text{for } x \cdot \nu \leq -B. \quad (4.13)$$

Moreover using (4.1)-(4.3) and (4.9)-(4.11), we obtain the following Cauchy problem for the residual $w_s(x, t, \nu)$ of the expansion (4.9)

$$n^2(x) \partial_t^2 w_r - \Delta w_r = F_r(x, t, \nu), \quad (4.14)$$

$$w_r|_{t < -B} = 0, \quad (4.15)$$

$$F_r(x, t, \nu) = (\Delta a_r(x, \nu)) \theta_r(t - \varphi(x, \nu)). \quad (4.16)$$

We now construct solutions of equations (4.10) with the Cauchy data (4.11)-(4.13). We rewrite equations (3.6) of the geodesic line $\Gamma(x, \nu)$ with the vector $p(x, \nu) = \nabla \varphi(x, \nu)$ in the following form

$$\frac{dx}{d\tau} = n^{-2}(x) p(x, \nu), \quad \frac{dp(x, \nu)}{d\tau} = \nabla (\ln n(x)), \quad (4.17)$$

where τ is the arc Riemannian length with the element length given by the formula $d\tau = n(\xi) |d\xi|$. Hence, we have along the geodesic line $\Gamma(x, \nu)$

$$2\nabla A(x, \nu) \cdot \nabla \varphi(x, \nu) = 2\nabla A(x, \nu) \cdot p(x, \nu) = 2n^2(x) \nabla A(x, \nu) \cdot \frac{dx}{d\tau} = 2n^2(x) \frac{dA(x, \nu)}{d\tau}.$$

Hence, the first equation (4.10) becomes

$$2n^2(x) \frac{dA(x, \nu)}{d\tau} + A(x, \nu) \Delta \varphi(x, \nu) = 0. \quad (4.18)$$

The solution of this equation with the initial condition (4.11) is given by (4.6).

Then equations for $a_k(x, \nu)$, $k \geq 0$, can be rewritten in the form

$$\frac{d}{d\tau} \left(\frac{a_k(x, \nu)}{A(x, \nu)} \right) = \frac{\Delta a_{k-1}(\xi, \nu)}{2n^2(x) A(x, \nu)}, \quad k \geq 0, \quad (4.19)$$

where we should formally have $a_{-1}(x, \nu) = A(x, \nu)$. It is easy to verify (4.19), if expressing $\Delta \varphi(x, \nu)$ via $A(x, \nu)$ using (4.18).

Taking into account the initial data (4.13), we obtain

$$a_k(x, \nu) = A(x, \nu) \int_{\Gamma(x, \nu)} \frac{\Delta a_{k-1}(\xi, \nu)}{2n^2(\xi)A(\xi, \nu)} d\tau, \quad k \geq 0, \quad x \cdot \nu \geq -B. \quad (4.20)$$

Let $m > 1$ be a sufficiently large integer which we will choose below. If $n(x) \in C^m(\mathbb{R}^3)$, then we have

$$\varphi(x, \nu) \in C^m(\mathbb{R}^3), A(x, \nu) \in C^{m-2}(\mathbb{R}^3), a_k(x, \nu) \in C^{m-4-2k}(\mathbb{R}^3), a_r(x, \nu) \in C^{m-4-2r}(\mathbb{R}^3).$$

Hence, the function $F_r(x, t, \nu) \in C^l(D(\nu, T))$, where $l = \min(m - 6 - 2r, r)$. By (4.13) and (4.16)

$$F_r(x, t, \nu) = 0 \text{ for } t \leq \varphi(x, \nu) \text{ and for } x \cdot \nu < -B. \quad (4.21)$$

We now prove that

$$w_r(x, t, \nu) = 0 \text{ for } t \in [-B, \varphi(x, \nu)]. \quad (4.22)$$

Indeed, (4.15) implies that (4.22) is true for $t \leq -B$. Let now (x^0, t^0) be an arbitrary point of the domain $Q(\nu) = \{(x, t) \in R^4 : -B < t < \varphi(x, \nu)\}$. Let $\tau(x, x^0)$ be the Riemannian distance between points x and x^0 . Denote

$$K(\nu, x^0) = \{(x, t) \in \mathbb{R}^4 \mid -B \leq t \leq \varphi(x^0, \nu) - \tau(x, x^0)\}.$$

Hence, $K(\nu, x^0)$ is the inner part of the characteristic cone with the vertex at the point $(x^0, \varphi(x^0, \nu))$ and bounded by the plane $t = -B$. Obviously $(x^0, t^0) \in K(\nu, x^0)$. By (4.21) $F_r(x, t, \nu) = 0$ for $(x, t) \in K(\nu, x^0)$. Apply now the method of energy estimates to the problem (4.14), (4.15) in the domain $K(\nu, x^0)$. Observe that the resulting surface integral will be non-negative on the lateral boundary $t = \varphi(x^0, \nu) - \tau(x, x^0)$ of the characteristic cone $K(\nu, x^0)$. Also, the integral over $\{t = -B\} \cap \overline{K}(\nu, x^0)$ will be zero due to (4.15). Hence, we obtain $w_r(x, t, \nu) \equiv 0$ for all $(x, t) \in K(\nu, x^0)$. Thus, $w_r(x^0, t^0, \nu) = 0$. Since (x^0, t^0) is an arbitrary point of the domain $Q(\nu)$, then we conclude that (4.22) holds.

Furthermore, using theorems 3.2, 4.1, corollary 4.2 and energy estimates of Chapter 4 of [23], one can easily prove that there exists unique solution w_r of equation (4.14), (4.15) such that

$$w_r(x, -B, \nu) = \partial_t w_r(x, -B, \nu) = 0$$

and $w_r(x, t, \nu) \in H^{l+1}(Y(t, \nu, T))$, $\partial_t w_r(\cdot, t, \nu) \in H^l(Y(t, \nu, T))$ for all $t \in (-B, T)$, where $Y(t, \nu, T) = D(\nu, T) \cap \{t = \text{const}\}$. Choosing $m = 15$ and $r = 3$, we obtain $l = 3$. Thus, $w_3(x, t, \nu) \in H^4(Y(t, \nu, T))$ and $\partial_t w_3(x, t, \nu) \in H^3(Y(t, \nu, T))$. Therefore the embedding theorem implies that $w_3 \in C^2(D(\nu, T))$.

Setting

$$\hat{v}(x, t, \nu) = \sum_{k=0}^3 a_k(x, \nu) \frac{(t - \varphi(x, \nu))^k}{k!} + w_3(x, t, \nu),$$

we obtain (4.8) as well as the required smoothness $\hat{v}(x, t, \nu) \in C^2(D(\nu, T))$. \square

4.1 Connection with the problem (2.3)-(2.5)

Let $v(x, t, \nu)$ be the solution of the problem (4.1)-(4.3) which is guaranteed by Theorem 1. Fix an arbitrary bounded domain $\Phi \subset \mathbb{R}^3$. And consider the behavior of the functions $\partial_t^k v(x, t, \nu)$, $k = 0, 1, 2$ for $x \in \Phi$ and $t \rightarrow \infty$. We refer to Theorem 4 of Chapter 10 of the book [44] as well as to Remark 3 after that theorem. It follows from these results that these functions decay exponentially as $t \rightarrow \infty$ as long as $x \in \Phi$.

Consider the Fourier transform $V(x, k, \nu)$ of the function v ,

$$V(x, k, \nu) = \int_{-\infty}^{\infty} v(x, t, \nu) \exp(-ikt) dt, \quad x \in G. \quad (4.23)$$

Next, theorem 3.3 of [43] and theorem 6 of Chapter 9 of [44] guarantee that $V(x, k, \nu) = u(x, k, \nu)$, where the function $u(x, k, \nu)$ is the solution to the equation (2.3)-(2.4).

Using the representation (4.8), we obtain

$$u(x, k, \nu) = A(x, \nu) \exp(-ik\varphi(x, \nu)) + \int_{\varphi(x, \nu)}^{\infty} \widehat{v}(x, t, \nu) \exp(-ikt) dt, \quad x \in \Phi. \quad (4.24)$$

Denote

$$\partial_t^j \widehat{v}_+(x, \varphi(x, \nu), \nu) = \lim_{t \rightarrow \varphi^+(x, \nu)} \partial_t^j \widehat{v}(x, t, \nu), \quad j = 0, 1.$$

Integrating by parts in (4.24), we obtain for $x \in \Phi$

$$\begin{aligned} u(x, k, \nu) &= A(x, \nu) \exp(-ik\varphi(x, \nu)) + \frac{\exp(-ik\varphi(x, \nu))}{ik} \widehat{v}_+(x, \varphi(x, \nu), \nu) \\ &\quad + \frac{1}{ik} \int_{\varphi(x, \nu)}^{\infty} \widehat{v}_t(x, t, \nu) \exp(-ikt) dt. \end{aligned}$$

Hence,

$$u(x, k, \nu) = A(x, \nu) \exp(-ik\varphi(x, \nu)) + O\left(\frac{1}{k}\right), \quad k \rightarrow \infty, \forall x \in \Phi, \forall \nu \in S^2.$$

This and (2.4) imply that for all $x \in \overline{\Omega}$, $\nu \in S^2$

$$u_{sc}(x, k, \nu) = A(x, \nu) \exp(ik\varphi(x, \nu)) - \exp(ikx \cdot \nu) + O\left(\frac{1}{k}\right), \quad k \rightarrow \infty. \quad (4.25)$$

For every vector $\nu \in S^2$ denote $S^+(\nu) = \{x \in S : x \cdot \nu > 0\}$.

5 Approximate reconstruction of the function $u_{sc}(x, k, \nu)$ for all pairs $\nu \in S^2, x \in S^+(\nu)$

Ignoring the term $O(1/k)$ in (4.25) and using (2.6), we obtain the following approximate formula

$$f(x, k, \nu) = A^2(x, \nu) + 1 - 2A(x, \nu) \cos[k(\varphi(x, \nu) - x \cdot \nu)], \quad \forall \nu \in S^2, \forall x \in S^+(\nu). \quad (5.1)$$

We now fix the pair $\nu \in S^2, x \in S^+(\nu)$ and consider $f(x, k, \nu)$ as the function of k for $k \geq k_1$, where $k_1 \geq k_0$ is a sufficiently large number. It is possible to figure out whether or not $\varphi(x, \nu) = x \cdot \nu$. Indeed, it follows from (5.1) that $\varphi(x, \nu) = x \cdot \nu$ if and only if $f(x, k, \nu) = \text{const.}$ for $k \geq k_1$, i.e. if and only if $\partial_k f(x, k, \nu) = 0, \forall k \geq k_1$.

Suppose now that $f(x, k, \nu) \neq \text{const.}$ for $k \geq k_1$. It follows from (5.1) that there exists a number $k_2 \geq k_1$ such that

$$f(x, k_2, \nu) = \max_{k \geq k_1} f(x, k, \nu) = (A(x, \nu) + 1)^2.$$

In particular, we find from here the number $A(x, \nu)$,

$$A(x, \nu) = \sqrt{f(x, k_2, \nu)} - 1. \quad (5.2)$$

By (5.1) there exists a sequence $\{k_n(x, \nu)\}_{n=3}^\infty \subset (k_2(x, \nu), \infty)$ such that

$$f(x, k_n(x, \nu), \nu) = \max_{k \geq k_1} f(x, k, \nu) \text{ and } k_2 < k_3 < \dots < k_n < \dots$$

Hence,

$$k_3(x, \nu)(\varphi(x, \nu) - x \cdot \nu) = k_2(x, \nu)(\varphi(x, \nu) - x \cdot \nu) + 2\pi.$$

Thus, we approximate the number $\varphi(x, \nu)$,

$$\varphi(x, \nu) = x \cdot \nu + \frac{2\pi}{k_3(x, \nu) - k_2(x, \nu)}. \quad (5.3)$$

Thus, it follows from (4.25) that formulae (5.2) and (5.3) provide us with an approximation of the function $u_{sc}(x, k, \nu)$ for sufficiently large values of k and for all pairs $\nu \in S^2, x \in S^+(\nu)$. We have obtained this approximation only using the data (2.6) for our inverse problem. We use the word ‘‘approximation’’ here because we got (5.1) via ignoring the term $O(1/k)$ in (4.25). Next, we should reconstruct the function $\beta(x)$. This is done in section 7.

6 Linearization

We assume in sections 6 and 7 that

$$\|\beta\|_{C(\overline{\Omega})} \ll 1. \quad (6.1)$$

Use the linearization method for $\varphi(x, \nu)$ proposed in [24] (see chapter 3 in this book and [35, 37]). Then, we represent solution to problem (??) in the form

$$\varphi(x, \nu) = \varphi_0(x, \nu) + \varphi_1(x, \nu), \quad (6.2)$$

where $\varphi_0(x, \nu) = x \cdot \nu$. By (2.1) $n^2(x) = 1 + 2\beta(x) + \beta^2(x)$. By (6.1) the term $\beta^2(x)$ can be neglected in the latter expression. Substituting the representation (6.2) in (??) and neglecting terms $(\nabla \varphi_1)^2$ and β^2 , we find

$$\nabla_x \varphi_1(x, \nu) \cdot \nabla_x \varphi_0(x, \nu) = \beta(x), \quad \varphi_1(x, \nu) = 0 \text{ for all } x \cdot \nu \leq -B. \quad (6.3)$$

Since $\nabla_x \varphi_0(x, \nu) = \nu$, then the left hand side of equation (6.3) is the derivative in the direction ν . Hence,

$$\varphi_1(x, \nu) = \int_{-x \cdot \nu - B}^0 \beta(x + \nu \sigma) d\sigma, \quad (6.4)$$

The right hand side of (6.4) is the integral over the segment $l(x, \nu)$ of a straight line and σ is its arc length. The segment $l(x, \nu)$ is stretched from the point x in the direction $-\nu$ and continues until reaching the point $\bar{x}(x, \nu) = x - (x \cdot \nu + B)\nu \in \Sigma(\nu)$. Note that by (3.3) $\bar{x}(x, \nu) \notin \Omega$.

Let $\nu \in S^2$ and $x \in S^+(\nu)$. Denote by $y = y(x, \nu) = x - 2(x \cdot \nu)\nu$ the second intersection point of the straight line $\xi = x + \nu \sigma$ with S . Let $L(x, \nu)$ denotes the segment of that straight line connecting points x and $y(x, \nu)$. Since the function $\beta(x) = 0$ outside of the ball Ω and since $S = \partial\Omega$, then equation (6.4) is equivalent with

$$\varphi_1(x, \nu) = \int_{L(x, \nu)} \beta(\xi) d\sigma, \forall \nu \in S^2, \forall x \in S^+(\nu). \quad (6.5)$$

7 Reconstructions

Formula (6.5) enables us to use two reconstruction procedures for finding the function $\beta(x)$. The first one is the inverse 2-d Radon transform, and the second one solves integral equations of the Abel kind. As it was pointed out in Introduction, the idea of second procedure goes back to the paper [6]. In both cases we can reconstruct the function $\beta(x)$ separately in each 2-d cross section $\{x_3 = a = \text{const.}\}$ of the ball Ω .

For any number $a \in \mathbb{R}$ consider the plane $P_a = \{x_3 = a\}$. Consider the disk $Q_a = \Omega \cap P_a$ and let the circle $S_a = S \cap P_a$ be its boundary. Then the radius of this circle is $B_a = \sqrt{B^2 - a^2}$. Clearly $Q_a \neq \emptyset$ for $a \in (-B, B)$ and $Q_a = \emptyset$ for $|a| \geq B$. Denote $0_a = (0, 0, a) \in Q_a$ the orthogonal projection of the origin on the plane P_a . We have

$$\Omega = \bigcup_{a=-B}^B Q_a, \partial\Omega := S = \bigcup_{a=-B}^B S_a.$$

Denote

$$\begin{aligned} S_0^2 &= \{\nu = (\nu_1, \nu_2, \nu_3) \in S^2 : \nu_3 = 0\}, \\ S_a^+(\nu) &= \{x \in S_a : x \cdot \nu > 0\}, \forall \nu \in S_0^2. \end{aligned}$$

7.1 Reconstruction using the inverse Radon transform

In this subsection we present the reconstruction formula based on the inverse 2-d Radon transform. First, we parameterize $L(x, \nu)$ in the conventional way of the parametrization of the Radon transform [26]. For $\nu \in S_0^2, x \in S_a^+(\nu)$, let m be the unit normal vector to the line $L(x, \nu)$ lying in the plane P_a and pointing outside of the point 0_a . Since we work now only with the plane P_a , we discard the third component. Let $\alpha \in (0, 2\pi]$ be the angle between m and the x_1 -axis. Then $m = m(\alpha) = (\cos \alpha, \sin \alpha)$. Let d be the signed

distance between $L(x, \nu)$ and the point 0_a (page 9 of [26]). It is clear that there exists a one-to-one correspondence between pairs (x, ν) and $(m(\alpha), d)$, $d(x, \nu) \in (-B_a, B_a)$,

$$(x, \nu) \Leftrightarrow (m(\alpha), d); \nu \in S_0^2, x \in S_a^+(\nu); \alpha = \alpha(x, \nu) \in (0, 2\pi], d = d(x, \nu). \quad (7.1)$$

Hence, we can write

$$L(x, \nu) = \{z_a = (z_1, z_2, a) \in Q_a : z \cdot m(\alpha) = d\}, \quad (7.2)$$

where $z = (z_1, z_2) \in \mathbb{R}^2$ and parameters $\alpha = \alpha(x, \nu)$ and $d = d(x, \nu)$ are defined as in (7.1).

Consider an arbitrary function $q = q(z) \in C^2(P_a)$ such that $q(z) = 0$ for $z \in P_a \setminus Q_a$. Hence,

$$\int_{L(x, \nu)} q(z) d\sigma = \int_{z \cdot m(\alpha) = d} q(z) d\sigma, \quad \forall x \in S_a(\nu), \quad \forall \nu \in S_0^2, \quad (7.3)$$

where $\alpha = \alpha(x, \nu)$, $s = s(x, \nu)$ are as in (7.1). In (7.3) σ is the arc length and the parametrization of $L(x, \nu)$ is given in (7.2). Therefore, using (7.1)-(7.3), we can define the 2-d Radon transform Rq of the function q as

$$(Rq)(x, \nu) = (Rq)(\alpha, d) = \int_{z \cdot m(\alpha) = d} g(z) d\sigma. \quad (7.4)$$

Let R^{-1} be the operator which is the inverse to the operator R in (7.4). The specific form of R^{-1} is well known, see, e.g. [26]. Hence, we do not present this form here for brevity. Then formulae (6.5) and (7.4) imply that

$$\beta(z, a) = R^{-1}(\varphi_1(x, \nu))(z, a), \quad \forall a \in (-B, B), \quad \forall z \in Q_a. \quad (7.5)$$

By (5.3) and (6.2) the function $\varphi_1(x, \nu)$ is approximately known for all $\nu \in S^2$ and for all $x \in S_a^+(\nu)$. Thus, formula (7.5) completes our reconstruction process via the inverse Radon transform.

7.2 Reconstruction via solutions of integral equations of the Abel type

Consider now again an arbitrary number $a \in (-B, B)$ and an arbitrary pair $\nu \in S_0^2, x \in S_a^+(\nu)$. Then the point $y(x, \nu) = (y_1, y_2, a) \in S_a$. Given the above pair x, ν , there exists unique point $y(x, \nu) \in S_a$. And vice versa: given a point $y \in S_a$ and a vector $\nu \in S_0^2$, there exists unique point $x \in S_a^+(\nu)$ such that the segment $L(x, \nu)$ passes through points x and y . Hence, we denote below $M(x, y)$ the segment of the straight line passing through points $x, y \in S_a$. Clearly the set $\{M(x, y)\}_{x, y \in S_a}$ coincides with the set $\{L(x, \nu)\}_{\nu \in S_0^2, x \in S_a^+(\nu)}$, i.e. $\{M(x, y)\}_{x, y \in S_a} = \{L(x, \nu)\}_{\nu \in S_0^2, x \in S_a^+(\nu)}$. Keeping this in mind, we rewrite (6.5) as

$$\psi(x, y) = \int_{M(x, y)} \beta(\xi) d\sigma, \quad \forall x, y \in S_a, \quad \forall a \in (-B, B), \quad (7.6)$$

where the function $\psi(x, y)$ is constructed from the function $\varphi_1(x, \nu)$ in an obvious manner, i.e. for each pair $x, y \in S_a$ we find the vector $\nu \in S_0^2$ such that $x \in S_a^+(\nu)$ and $y = y(x, \nu)$, and then set $\psi(x, y) = \psi(x, y(x, \nu)) = \varphi_1(x, \nu)$.

In the plane P_a we introduce polar coordinates r, φ of the variable point $\xi = (\xi_1, \xi_2)$ as $\xi_1 = r \cos \phi, \xi_2 = r \sin \phi$. We characterize $M(x, y)$ by the polar coordinates (ρ, α) of its middle point $(x + y)/2$. Hence, $|x - y| = 2\sqrt{B^2 - a^2 - \rho^2}$. Change variables in the integral (7.6) as

$$\sigma \Longleftrightarrow r, \sigma = \sqrt{B^2 - a^2 - \rho^2} - \sqrt{r^2 - \rho^2},$$

Then $d\sigma = -rdr/\sqrt{r^2 - \rho^2}$. The equation of $M(x, y)$ can be rewritten as

$$\phi = \alpha + (-1)^j \arccos \frac{\rho}{r}, \quad j = 1, 2, \quad r \geq \rho, \quad (7.7)$$

where $j = 1$ corresponds to the part of the segment $M(x, y)$ between points $(x + y)/2$ and x , and $j = 2$ to the rest of $M(x, y)$.

Obviously, there exists a one-to-one correspondence, up to the symmetry mapping $(x, y) \Leftrightarrow (y, x)$ between pairs $x, y \in S_a$ and $(\rho, \alpha) \in (0, R) \times (0, 2\pi)$. Denote

$$\beta(r \cos \phi, r \sin \phi, a) = \tilde{\beta}(r, \varphi, a) \text{ and } \tilde{\psi}(\rho, \alpha, a) = \psi(x, y).$$

Using (7.7), we rewrite equation (7.6) as

$$\sum_{j=1}^2 \int_{\rho}^{B_a} \tilde{\beta}(r, \alpha + (-1)^j \arccos \frac{\rho}{r}, a) \frac{r}{\sqrt{r^2 - \rho^2}} dr = \tilde{\psi}(\rho, \alpha, z). \quad (7.8)$$

Represent functions $\tilde{\beta}(r, \varphi, a)$ and $\tilde{\psi}(\rho, \alpha, a)$ via Fourier series,

$$\begin{aligned} \tilde{\beta}(r, \varphi, a) &= \sum_{n=-\infty}^{\infty} \tilde{\beta}_n(r, a) \exp(in\phi), \\ \tilde{\psi}(\rho, \alpha, z) &= \sum_{n=-\infty}^{\infty} \tilde{\psi}_n(\rho, a) \exp(in\alpha). \end{aligned}$$

Multiplying both sides of (7.8) by $\exp(-in\alpha)/(2\pi)$ and integrating with respect to $\alpha \in (0, 2\pi)$, we obtain for all $n = 0, \pm 1, \pm 2, \dots$

$$\int_{\rho}^{B_a} \tilde{\beta}_n(r, a) \cos\left(n \arccos \frac{\rho}{r}\right) \frac{r dr}{\sqrt{r^2 - \rho^2}} = \tilde{\psi}_n(\rho, a), \quad \rho \in (0, B_a) \quad (7.9)$$

This is the integral equation of the Abel type. To solve equation (7.9), we apply first the operator A to both sides of (7.9), where

$$A(h(\rho))(\omega) = \frac{1}{\pi} \int_s^{B_a} \frac{h(\rho) \rho d\rho}{\sqrt{\rho^2 - \omega^2}}, \quad \omega \in (0, B_a).$$

Changing the limits of the integration, we obtain

$$\begin{aligned} \frac{1}{\pi} \int_{\omega}^{B_a} \tilde{\beta}_n(r, a) \left[\int_{\omega}^r \frac{\rho}{\sqrt{\rho^2 - \omega^2} \cdot \sqrt{r^2 - \rho^2}} \cos \left(n \arccos \frac{\rho}{r} \right) d\rho \right] dr \\ = A \left(\tilde{\psi}_n(\rho, a) \right) (\omega). \end{aligned} \quad (7.10)$$

Change variables in the inner integral (7.10) as

$$\rho \Leftrightarrow \theta, \rho^2 = \omega^2 \cos^2(\theta/2) + r^2 \sin^2(\theta/2).$$

Then

$$\begin{aligned} 2\rho d\rho &= (r^2 - \omega^2) \sin \theta \cos \theta d\theta, \\ \sqrt{\rho^2 - \omega^2} \cdot \sqrt{r^2 - \rho^2} &= (r^2 - \omega^2) \sin \theta \cos \theta. \end{aligned}$$

Hence, equation (7.10) can be rewritten as

$$\begin{aligned} \int_{\omega}^{B_a} \tilde{\beta}_n(r, a) Q_n(r, \omega) dr &= 2A \left(\tilde{\psi}_n(\rho, a) \right) (\omega), \\ Q_n(r, \omega) &= \frac{1}{\pi} \int_0^{\pi} \cos \left(n \arccos \frac{\sqrt{r^2 \cos^2(\theta/2) + \omega^2 \sin^2(\theta/2)}}{r} \right) d\theta. \end{aligned} \quad (7.11)$$

We have $Q_n(\omega, \omega) = 1$. Hence, differentiating (7.11) with respect to ω , we obtain Volterra integral equation of the second kind

$$\tilde{\beta}_n(\omega, a) - \int_{\omega}^{B_a} \tilde{\beta}_n(r, a) T_n(r, \omega) dr = -\frac{\partial}{\partial \omega} \left[2A \left(\tilde{\psi}_n(\rho, z) \right) (\omega) \right], \quad \omega \in (0, B_a), \quad (7.12)$$

$$T_n(r, \omega) = \frac{n}{\pi \sqrt{r^2 - \omega^2}} \times \quad (7.13)$$

$$\int_0^{\pi} \left[\sin \left(n \arccos \left(\frac{\sqrt{r^2 \cos^2(\theta/2) + \omega^2 \sin^2(\theta/2)}}{r} \right) \right) \frac{\sin(\theta/2)}{\sqrt{r^2 \cos^2(\theta/2) + \omega^2 \sin^2(\theta/2)}} \right] d\theta.$$

It follows from (7.13) that the kernel of integral equation (7.12) has the form

$$T_n(r, \omega) = \frac{\tilde{T}_n(r, \omega)}{\sqrt{r^2 - \omega^2}},$$

where the function $\tilde{T}_n(r, \omega)$ is continuous for $0 \leq \omega \leq r \leq B_a$. Therefore, it follows from the theory of Volterra integral equations of the second kind that for each $a \in (-B, B)$ there exists a solution $\tilde{\beta}_n(\omega, a) \in C[0, B_a]$ of equation (7.12) and this solution is unique. Furthermore, it is well known from that theory that equation (7.12) can be solved iteratively as

$$\tilde{\beta}_n^0(\omega, a) = -\frac{\partial}{\partial \omega} \left[M \left(\tilde{\psi}_n(\rho, a) \right) (\omega) \right], \quad (7.14)$$

$$\tilde{\beta}_n^k(\omega, a) = \int_{\omega}^{\rho_0} \tilde{\beta}_n^{k-1}(r, a) T_n(r, \omega) dr - \frac{\partial}{\partial \omega} \left[M \left(\tilde{\psi}_n(\rho, a) \right) (\omega) \right], k = 1, 2, \dots \quad (7.15)$$

and this process converges in the space $C[0, \rho_0]$ to the solution $\tilde{\beta}_n(\omega, a)$ of equation (7.12). Formulae (7.14) and (7.15) finish our second reconstruction procedure.

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